Quasi-coordinates from the point of view of Lie algebroid structures

2007 J. Phys. A: Math. Theor. 40 10031

(http://iopscience.iop.org/1751-8121/40/33/008)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 10031–10048 doi:10.1088/1751-8113/40/33/008

Quasi-coordinates from the point of view of Lie algebroid structures

José F Cariñena[1], **Joana M Nunes da Costa**[2] **and Patrícia S Santos**[3]

[1] Department of Theoretical Physics, University of Zaragoza, Spain
[2] CMUC, Department of Mathematics, University of Coimbra, Portugal
[3] CMUC, University of Coimbra and Engineering Institute of Coimbra, Portugal

Received 7 December 2006, in final form 8 June 2007
Published 1 August 2007
Online at stacks.iop.org/JPhysA/40/10031

Abstract

A geometrical description of the Lagrangian dynamics in quasi-coordinates on the tangent bundle, using the Lie algebroid framework, is given. Linear non-holonomic systems on Lie algebroids are solved in local coordinates adapted to the constraints, through Lagrangian multipliers and Gibbs–Appell generalized methods.

PACS numbers: 02.20.Sv, 02.40.−k, 45.20.Tj
Mathematics Subject Classification: 17B45, 70G10, 70H03, 70S05

1. Introduction

It is well known that the use of quasi-coordinates has many applications in physics and engineering (see e.g. [23]). As far as we know there is no systematic treatment of the concept of quasi-coordinates from the geometric point of view. The aim of this paper is to develop the geometric approach to such a concept in the Lie algebroid framework and to prove the efficiency of this geometric tool for solving different problems.

The dynamics of a classical system is usually described in terms of (local) generalized coordinates on a manifold Q, the configuration space of the system, which give rise to a particular type of coordinates on its tangent bundle TQ, i.e. positions q^i and their velocities $v^i \equiv \dot{q}^i$. However, there are some cases where it is useful to consider a different set of coordinates on TQ. For example, Euler equations for the rigid body are written in terms of the three Euler angles (θ, ψ, φ) and the three components of the angular velocity $(\omega_\theta, \omega_\psi, \omega_\varphi)$ instead of the velocities $(\dot{\theta}, \dot{\psi}, \dot{\varphi})$. These new kind of coordinates are called quasi-velocities. Although the calculations in quasi-coordinates are difficult, they are often used to solve many types of mechanical systems (see [14, 23, 29]). The geometrical use of quasi-coordinates amounts 'to forget' the tangent structure of the tangent bundle and to use only the vector bundle structure and the Lie algebra structure on the set of vector fields, that is to say, not using the usual tangent bundle coordinates associated with a coordinate system on the base manifold. This geometrical framework leads naturally to the use of Lie algebroids and to the development of the Lagrangian mechanics on Lie algebroids.

Lie algebroids provide a very general framework for Mechanics, including mechanical systems with symmetries. Roughly speaking, a Lie algebroid is a generalization of both a Lie algebra and a tangent bundle structure, both being the simplest examples of Lie algebroids. Moreover, the Lie algebroid structure is well adapted to variational calculus for constrained systems and may appear in the reduction process.

Lie algebroids were introduced by Pradines [24] as infinitesimal objects corresponding to Lie groupoids. In the last years several authors have studied the theory of Lie algebroids giving important contributions for the knowledge of their properties and applications. Among others, Higgins *et al* [13] introduced the notion of prolongation of a Lie algebroid over a map; Weinstein [28] was the first to study Lagrangian mechanics on Lie algebroids and Martínez [19] developed the formalism for the Lagrangian mechanics on Lie algebroids, generalizing the fundamental geometrical elements of Lagrangian mechanics (see also [16] and references therein). Unfortunately, this structure of Lie algebroid is not well known for most physicists and however it may play a very relevant role in many physical problems, for instance in BRST formalism, Yang–Mills and topological field theory [15, 22, 27].

One of the aims of this paper is to prove the efficiency of Lie algebroid structures to deal with systems with linear non-holonomic constraints, providing us with a generalized version of the Lagrangian multipliers [1, 2, 25] and Gibbs–Appell [11, 17] methods. The solutions of these systems are obtained in local coordinates adapted to the constraints. These adapted coordinates on a Lie algebroid play the role of quasi-coordinates on a tangent bundle. For the non-holonomic systems in a Lie algebroid we refer the first paper to the subject by Cortés *et al* [6] and the papers of Mestdag *et al* [20, 21], Cortés *et al* [7] and Cariñena *et al* [5].

This paper is organized in the following way. The geometric approach to solve classical systems using quasi-coordinates is developed in section 2. A brief introduction to Lie algebroids is given in section 3, and the geometric approach to quasi-coordinates is analysed from the Lie algebroid structure point of view in section 4. For the self-containedness of the paper, general changes of coordinates on a Lie algebroid are studied in section 5. In the last section we apply the above formalism to solve systems with linear non-holonomic constraints in Lie algebroids, using Lagrange multipliers and Gibbs–Appell methods in the Lie algebroid framework. At this point, the advantage of using adapted coordinates is clear.

2. A geometric approach to quasi-coordinates

The use of quasi-coordinates has been shown to be very efficient in describing the motion of some dynamical systems. For instance, the area swept by the line joining a planet with the sun for the motion of the planet, or the components of the angular momentum for describing the motion of a rigid body with a fixed point are quasi-coordinates. As pointed out in [11, 29], the configuration of a dynamical system cannot be in general described by quasi-coordinates, but it is possible to describe the displacement by using quasi-coordinates, more specifically quasi-velocities. Next, we explain the geometric meaning of such quasi-velocities.

Let $\pi_Q : T^*Q \to Q$ denote the cotangent bundle of an n-dimensional differentiable manifold Q. It is well known that a local 1-form γ on an open set U of Q, i.e. a section for π_Q over U, defines a linear function $\widehat{\gamma} \in C^\infty(\mathcal{U})$, where $\mathcal{U} = \tau_Q^{-1}(U)$ for $\tau_Q : TQ \to Q$, as follows:

$$\widehat{\gamma}(v) = \langle \gamma_{\tau_Q(v)}, v \rangle, \qquad v \in \mathcal{U},$$

for a vector v in a point of the open set U.

The choice of a local chart on an open set U of Q, with coordinates $(q^1, \ldots, q^n)$, determines a basis $(\partial/\partial q^1, \ldots, \partial/\partial q^n)$ of the tangent space at each point of $\mathcal{U} = \tau_Q^{-1}(U)$ and defines an associated local coordinate system on TQ. The $2n$ coordinates are now the n basic functions q^i (or in a more rigourous notation $q^i \circ \tau_Q$) and the corresponding functions on the fibres $v^i = \widehat{\mathrm{d}q^i}$. Note that $\mathrm{d}q^1 \wedge \cdots \wedge \mathrm{d}q^n \neq 0$, and therefore the functions $\widehat{\mathrm{d}q^i}$, for $i = 1, \ldots, n$, are functionally independent. When the local expression of γ in terms of local coordinates on U is $\gamma = \gamma_i(q)\,\mathrm{d}q^i$, then $\widehat{\gamma}$ is the function $\widehat{\gamma}(v) = \gamma_i(\tau_Q(v))v^i$.

Instead of using $\{\mathrm{d}q^1, \ldots, \mathrm{d}q^n\}$ to define a chart on $\mathcal{U}$, we can alternatively make use, together with the base coordinate functions, of any other set of n 1-forms $\{\alpha^1, \ldots, \alpha^n\}$, given locally by

$$\alpha^i = \alpha^i_j(q)\,\mathrm{d}q^j, \qquad \forall\, i = 1, \ldots, n,$$

with the only condition of being linearly independent at each point, i.e. $\alpha^1 \wedge \cdots \wedge \alpha^n \neq 0$. In this case the $2n$ coordinates on $\mathcal{U}$ are the n basic functions q^i together with the linear functions $\{\widehat{\alpha^1}, \ldots, \widehat{\alpha^n}\}$. These new coordinates on the fibres $\{\widehat{\alpha^1}, \ldots, \widehat{\alpha^n}\}$ are but linear combinations, with basic functions as coefficients, of the usual velocities. Note that the 1-forms α^i do not need to be exact, but in the case of all of them being exact, the functions $\widehat{\alpha^i}$ would be the velocities corresponding to a new coordinate system on the base manifold Q. The new fibre coordinates $w^i = \widehat{\alpha^i}$, which play the role of velocities, are called *quasi-velocities*, while the coordinates (q^i, w^i) are called *quasi-coordinates* on TQ. Note that in some cases, for instance when Q is a Lie group G, we can globally define quasi-velocities on Q while velocities can only be defined locally.

The fact that $\alpha^1 \wedge \cdots \wedge \alpha^n \neq 0$ points out that there exist functions $\beta^i_j(q)$ such that

$$\mathrm{d}q^i = \beta^i_j(q)\alpha^j, \qquad \forall\, i = 1, \ldots, n,$$

with $\det\big(\beta^i_j\big) \neq 0$. The matrix with entries $\beta^i_j(q)$ will be the inverse matrix of $\big(\alpha^i_j(q)\big)$, i.e.

$$\beta^i_j(q)\alpha^j_k(q) = \delta_{ik},$$

for all $q \in U$. The quasi-velocities w^i are associated with a basis of vector fields $\{X_1, \ldots, X_n\}$ on Q, dual to the basis of 1-forms $\{\alpha^1, \ldots, \alpha^n\}$, i.e. $\langle \alpha^i, X_j \rangle = \delta_{ij}$. Then locally, in quasi-coordinates,

$$X_j = \beta^i_j \frac{\partial}{\partial q^i}. \tag{1}$$

The partial derivative $\partial/\partial q^i$ in (1) is given, in the usual coordinates, by

$$\frac{\partial}{\partial q^i} = \left.\frac{\partial}{\partial q^i}\right|_{v\,\mathrm{const}} + \frac{\partial \beta^k_l}{\partial q^i}\alpha^l_r v^r \frac{\partial}{\partial v^k}\,. \tag{2}$$

The relations among fibre coordinates are

$$w^i = \alpha^i_j v^j, \qquad v^i = \beta^i_j w^j,$$

consequently,

$$\frac{\partial w^i}{\partial v^j} = \alpha^i_j, \qquad \frac{\partial v^i}{\partial w^j} = \beta^i_j$$

and

$$\frac{\partial}{\partial v^i} = \alpha^j_i \frac{\partial}{\partial w^j}, \qquad \frac{\partial}{\partial w^i} = \beta^j_i \frac{\partial}{\partial v^j}.$$

The local expressions of the Liouville vector field Δ and the vertical endomorphism S (see [8–10]) in terms of quasi-coordinates are, respectively, given by

$$\Delta = v^i \frac{\partial}{\partial v^i} = \beta^i_j w^j \alpha^k_i \frac{\partial}{\partial w^k} = w^j \frac{\partial}{\partial w^j}$$

and

$$S = \frac{\partial}{\partial v^i} \otimes \mathrm{d}q^i = \alpha^j_i \frac{\partial}{\partial w^j} \otimes \mathrm{d}q^i = \frac{\partial}{\partial w^j} \otimes \alpha^j.$$

The explicit coordinate expression of a second-order differential equation (SODE) vector field $D \in \mathfrak{X}(TQ)$ in terms of quasi-coordinates is

$$D = \beta^i_k(q) w^k \frac{\partial}{\partial q^i} + f^i(q, w) \frac{\partial}{\partial w^i}.$$

In fact, if

$$D = h^i(q, w) \frac{\partial}{\partial q^i} + f^i(q, w) \frac{\partial}{\partial w^i},$$

then

$$S(D) = \alpha^j_i h^i(q, w) \frac{\partial}{\partial w^j},$$

and therefore (by the definition of a SODE vector field) $S(D) = \Delta$ if and only if $\alpha^j_i h^i(q, w) = w^j$, or equivalently, $h^i = \beta^i_j w^j$.

Let us consider a regular Lagrangian system characterized by a function $\mathcal{L} \in C^\infty(TQ)$ and non-conservative force 1-form $\mathcal{Q}$ locally defined by $\mathcal{Q}(q, v) = Q_i(q, v)\, \mathrm{d}q^i$. The energy of the system in the absence of non-conservative forces is given by

$$E_\mathcal{L} = \Delta\mathcal{L} - \mathcal{L} = w^i \frac{\partial \mathcal{L}}{\partial w^i} - \mathcal{L}.$$

The Cartan 1-form $\theta_\mathcal{L} = d\mathcal{L} \circ S$ is given in quasi-coordinates by

$$\theta_\mathcal{L} = \left(\frac{\partial \mathcal{L}}{\partial q^k} \mathrm{d}q^k + \frac{\partial \mathcal{L}}{\partial w^i} \mathrm{d}w^i \right) \circ \left(\alpha^j_i \frac{\partial}{\partial w^j} \otimes \mathrm{d}q^i \right) = \alpha^j_i \frac{\partial \mathcal{L}}{\partial w^j} \mathrm{d}q^i = \frac{\partial \mathcal{L}}{\partial w^j} \alpha^j,$$

and consequently, the Cartan 2-form $\omega_\mathcal{L}$ defined by $\omega_\mathcal{L} = -\mathrm{d}\theta_\mathcal{L}$, turns out to be

$$\omega_\mathcal{L} = \frac{1}{2} \left[\left(\frac{\partial \alpha^k_i}{\partial q^j} - \frac{\partial \alpha^k_j}{\partial q^i} \right) \frac{\partial \mathcal{L}}{\partial w^k} + \alpha^k_i \frac{\partial^2 \mathcal{L}}{\partial q^j \partial w^k} - \alpha^k_j \frac{\partial^2 \mathcal{L}}{\partial q^i \partial w^k} \right] \mathrm{d}q^i \wedge \mathrm{d}q^j$$
$$+ \alpha^k_i \frac{\partial^2 \mathcal{L}}{\partial w^j \partial w^k} \mathrm{d}q^i \wedge \mathrm{d}w^j,$$

that is,

$$\omega_\mathcal{L} = \frac{1}{2} \left[\gamma^k_{ml} \frac{\partial \mathcal{L}}{\partial w^k} + \beta^j_l \frac{\partial^2 \mathcal{L}}{\partial q^j \partial w^m} - \beta^i_m \frac{\partial^2 \mathcal{L}}{\partial q^i \partial w^l} \right] \alpha^m \wedge \alpha^l + \frac{\partial^2 \mathcal{L}}{\partial w^j \partial w^k} \alpha^k \wedge \mathrm{d}w^j,$$

where the functions γ^k_{ml} are given by

$$\gamma^k_{ml} = \beta^j_m \beta^i_l \left(\frac{\partial \alpha^k_j}{\partial q^i} - \frac{\partial \alpha^k_i}{\partial q^j} \right).$$

These functions are known in the literature as *Hamel symbols* (see [14, 23]).

Assuming that the Lagrangian $\mathcal{L}$ is regular, we can write the coordinate expression of the dynamical equation $\mathrm{i}(X)\omega_\mathcal{L} = \mathrm{d}E_\mathcal{L} - \mathcal{Q}$ in terms of quasi-coordinates. The dynamics will be given by a SODE vector field of the form $X = \beta^i_m w^m \partial/\partial q^i + f^m(q, w)\partial/\partial w^m$ (see [9]). The left-hand side of the dynamical equation becomes

$$\mathrm{i}(X)\omega_{\mathcal{L}} = \left(\gamma^k_{ml}\frac{\partial \mathcal{L}}{\partial w^k} + \beta^j_l \frac{\partial^2 \mathcal{L}}{\partial q^j \partial w^m} - \beta^i_m \frac{\partial^2 \mathcal{L}}{\partial q^i \partial w^l}\right) w^m \alpha^l + w^m \frac{\partial^2 \mathcal{L}}{\partial w^j \partial w^m}\,\mathrm{d}w^j - f^m \frac{\partial^2 \mathcal{L}}{\partial w^m \partial w^l}\alpha^l,$$

while the right-hand side is

$$\mathrm{d}E_{\mathcal{L}} - \mathcal{Q} = \left(w^k \beta^i_l \frac{\partial^2 \mathcal{L}}{\partial q^i \partial w^k} - \beta^i_l \frac{\partial \mathcal{L}}{\partial q^i}\right)\alpha^l + w^k \frac{\partial^2 \mathcal{L}}{\partial w^j w^k}\mathrm{d}w^j - \Upsilon_l \alpha^l,$$

where $\Upsilon_l = \beta^i_l Q_i$ is the l-component in quasi-coordinates of the external force $\mathcal{Q}$. Therefore,

$$w^m \gamma^k_{ml}\frac{\partial \mathcal{L}}{\partial w^k} - w^m \beta^i_m \frac{\partial^2 \mathcal{L}}{\partial q^i \partial w^l} - f^m \frac{\partial^2 \mathcal{L}}{\partial w^m \partial w^l} = -\beta^i_l \frac{\partial \mathcal{L}}{\partial q^i} - \Upsilon_l.$$

The dynamical equation is equivalent to $\pounds_X \theta_{\mathcal{L}} = \mathrm{d}\mathcal{L} + \Upsilon$, where $\pounds_X$ is the Lie derivative in the direction of X. In traditional physics notation, the above equation is equivalent to the following system of generalized Euler–Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial w^l}\right) = \beta^j_l \frac{\partial \mathcal{L}}{\partial q^j} + w^m \gamma^k_{ml}\frac{\partial \mathcal{L}}{\partial w^k} + \Upsilon_l. \tag{3}$$

When the Lagrangian $\mathcal{L}$ is of mechanical type (i.e. $\mathcal{L} = T - V$, with T the kinetic energy and V the potential energy), the equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial w^l}\right) = \beta^j_l \frac{\partial T}{\partial q^j} + w^m \gamma^k_{ml}\frac{\partial T}{\partial w^k} + \Pi_l, \tag{4}$$

where $\Pi_l = \beta^j_l F_j$ is the l-component in quasi-coordinates of the external force $\mathcal{F} = -\mathrm{d}V + \mathcal{Q}$ (see [11, 29]).

3. Lie algebroids

As was already mentioned in the introduction the tangent bundle to a manifold can be viewed as a particular example of a more general structure, that of a Lie algebroid. This structure has been shown to be of a great usefulness in mechanics since the pioneer paper by Weinstein [28]. In particular, Martínez showed in [19] that the Lagrangian theory can be developed directly in the Lie algebroid formalism by using new geometric tools which generalize the vertical endomorphism and the Liouville vector field.

As this structure is of a recent use in physics and we think that it is not well known for most physicists, we summarize in this section the basic concepts and definitions of the theory of Lie algebroids.

Definition 3.1. *A Lie algebroid with base Q is a vector bundle $\tau_A : A \to Q$, together with a Lie algebra structure in the space of its sections given by a Lie product $[\cdot, \cdot]_A$, and a vector bundle map over the identity in the base, called the anchor, $\rho : A \to TQ$, inducing a map between the corresponding spaces of sections, to be denoted by the same name and symbol, such that*

$$[v, \varphi w]_A = \varphi[v, w]_A + (\rho(v)\varphi)w,$$

for any pair (v, w) of sections for τ_A and each $\varphi \in C^\infty(Q)$.

As a consequence of the definition, the morphism between sections induced by ρ, $\rho : \Gamma(A) \to \mathfrak{X}(Q)$, is a Lie algebra homomorphism.

Let $\{q^i | i = 1, \ldots, n\}$ be local coordinates in a chart on an open set $U \subset Q$, and let $\{e_\alpha | \alpha = 1, \ldots, s\}$ be a basis of local sections of the bundle $U_A = \tau_A^{-1}(U) \to Q$. Each local section v is written as $v = \mathbf{v}^\alpha e_\alpha$. The local coordinates of a point $p \in U_A$ such that $p = v(q)$ are $p = (q^i, \mathbf{v}^\alpha(\tau_A(p)))$. In a similar way, the corresponding dual basis $\{e^\alpha | \alpha = 1, \ldots, s\}$ of local sections on $U_{A^*} = \pi_A^{-1}(U)$ allows us to define local coordinates (q^i, μ_α) for the dual bundle $\pi_{A^*} : A^* \to Q$.

If the local expressions for the Lie product and the anchor map are

$$[e_\alpha, e_\beta]_A = c_{\alpha\beta}^\gamma e_\gamma, \qquad \rho(e_\alpha) = \rho_\alpha^i \frac{\partial}{\partial q^i}, \tag{5}$$

where $\alpha, \beta, \gamma = 1, \ldots, s$, and $i = 1, \ldots, n$, the functions $c_{\alpha\beta}^\gamma \in C^\infty(U)$ and $\rho_\alpha^i \in C^\infty(U)$ are called *structure functions of the Lie algebroid*. These are not arbitrary functions, they should satisfy the conditions for ρ to be a Lie algebra homomorphism, which are

$$\rho_\alpha^j \frac{\partial \rho_\beta^i}{\partial q^j} - \rho_\beta^j \frac{\partial \rho_\alpha^i}{\partial q^j} = \rho_\gamma^i c_{\alpha\beta}^\gamma, \qquad \forall\, i = 1, \ldots, n, \tag{6}$$

and those for the Leibniz condition and the Jacobi identity of the bracket $[\cdot, \cdot]_A$:

$$\sum_{\mathrm{cycl}(\alpha,\beta,\gamma)} \left(c_{\alpha\beta}^\nu c_{\nu\gamma}^\mu + \rho_\gamma^i \frac{\partial c_{\beta\alpha}^\mu}{\partial q^i} \right) = 0, \qquad \forall\, \mu = 1, \ldots, s. \tag{7}$$

Equations (5) and (6) are called *structure equations of the Lie algebroid*.

Examples of Lie algebroids are the tangent bundle of a manifold Q, with the identity as an anchor map and the usual bracket of vector fields, or any integrable sub-bundle of it, and also a finite-dimensional Lie algebra $\mathfrak{g}$, considered as a vector bundle over one point, for which the anchor vanishes identically and the bracket is that of $\mathfrak{g}$. In the first case, with the usual choice of coordinates (q^i, v^i) on $A = TQ$, induced from local coordinates (q^i) on the base Q, the structure functions are $c_{ij}^k = 0$ and $\rho_j^i = \delta_{ij}$. However, in arbitrary coordinates, the structure functions c_{ij}^k of the Lie algebroid $\tau_Q : TQ \to Q$ do not vanish. For the case of the Lie algebra $\mathfrak{g}$, the structure functions $c_{\alpha\beta}^\gamma$ are the structure constants of the Lie algebra and $\rho_\alpha^i = 0$.

Given a Lie algebroid $(A, \rho, [\cdot, \cdot]_A)$ over Q, there exists a graded derivation d_A of degree 1 of the graded exterior algebra of forms of A, $\Omega^\bullet(A)$, to be called A-forms, which is nilpotent of order 2, i.e. $d_A^2 = 0$. It is called the *exterior differential of the Lie algebroid*. In the particular cases of the tangent bundle TQ and that of a Lie algebra $\mathfrak{g}$, d_A reduces to the de Rham operator d on the manifold Q and the Chevalley–Eilenberg differential $d_{\mathfrak{g}}$, respectively.

4. Quasi-coordinates and the tangent bundle as a Lie algebroid

The choice of quasi-coordinates (q^i, w^i) on the tangent bundle TQ determines a local basis of sections for $\tau_{TQ} : T(TQ) \to TQ$, i.e. a local basis of vector fields on TQ. This basis is $\{X_j, \partial/\partial w^j | j = 1, \ldots, n\}$, with $X_j = \beta_j^i\, \partial/\partial q^i$ being an element of the dual basis of $\{\alpha^i | i = 1, \ldots, n\}$; $\{q_i | i = 1, \ldots, n\}$ is the set of local coordinates on Q.

The structure functions of the Lie algebroid $\tau_{TQ} : T(TQ) \to TQ$ with respect to such a local basis are given by

$$[X_r, X_l] = \gamma_{rl}^m X_m, \qquad \left[X_m, \frac{\partial}{\partial w^k}\right] = 0, \qquad \left[\frac{\partial}{\partial w^i}, \frac{\partial}{\partial w^j}\right] = 0,$$

$$\rho(X_m) = X_m, \qquad \rho\left(\frac{\partial}{\partial w^j}\right) = \frac{\partial}{\partial w^j}.$$

These relations give a geometric meaning to Hamel symbols introduced when dealing with quasi-coordinates on a tangent bundle. The Hamel symbols are not arbitrary, they must satisfy the structure equations (6) and (7).

Using the machinery of Lie algebroid theory, in particular the exterior differential, which in this case reduces to de Rham exterior differential on TQ, one can recover all the expressions obtained in a change from usual to quasi-coordinates [4]. This is the case, for instance, of the expressions in quasi-coordinates given in section 2: if the Lagrangian is regular and a non-conservative force $\mathcal{Q}$ is given, the dynamics equation $\mathrm{i}(X)\omega_{\mathcal{L}} = \mathrm{d}E_{\mathcal{L}} - \mathcal{Q}$ has a unique solution $X = w^m X_m + f^m \partial/\partial w^m$ that satisfies the generalized Euler–Lagrangian equations (3).

Example 4.1. Let us consider a particle P (mass $= 1$) moving in a plane under the action of a force of magnitude $F(r)$ on the direction of a fixed point O, where r represents the distance between the point O and the particle P. Let θ be the angle that the line OP makes with a fixed direction in the plane, and $\dot{A}$ the area swept by unity of time by the line. The configuration space of the system is $Q = \mathbb{R}^2 - \{(0,0)\}$ and the usual polar coordinates on $\mathbb{R}^2 - \{(x,0)\}$ extend to the velocity phase space TQ as $q^1 = r, q^2 = \theta, \dot{r}, \dot{\theta}$. In order to solve the equations of motion we use the following set of quasi-velocities on TQ:$w^1 = \dot{r}$, $w^2 = 2\dot{A} = r^2\dot{\theta}$. The matrices α and β are as follows:

$$\alpha = \left(\alpha^i_j\right) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, \qquad \beta = \left(\beta^i_j\right) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}.$$

The motion of the particle P is described by the regular Lagrangian

$$\mathcal{L} = \frac{1}{2}(w^1)^2 + \frac{1}{2}\frac{(w^2)^2}{r^2} - V(r),$$

where $F_1 = -\mathrm{d}V/\mathrm{d}r = F(r)$ and $F_2 = 0$. The equations of motion (4) are equivalent to

$$\begin{cases} \Pi_1(r,\theta,w^1,w^2) = \dot{w}^1 - \dfrac{(w^2)^2}{r^3} \\ \Pi_2(r,\theta,w^1,w^2) = \dfrac{\dot{w}^2}{r^2}. \end{cases}$$

Since $\Pi_1 = \beta^1_1 F_1 + \beta^2_1 F_2 = F_1$ and $\Pi_2 = \beta^1_2 F_1 + \beta^2_2 F_2 = 0$, the dynamics is given by the integral curves of the vector field $X = w^m X_m + f^i \partial/\partial w^i$ satisfying

$$\begin{cases} \dot{w}^1 = F(r) + \dfrac{(w^2)^2}{r^3} = f^1 \\ \dot{w}^2 = 0 = f^2, \end{cases}$$

where $\dot{r} = w^1$ and $\dot{\theta} = w^2/r^2$. As we can easily see, the quasi-velocity w^2 is a constant of motion, i.e. the area swept $\dot{A}$ is a constant of motion.

The usual coordinates on TQ are $q^1 = r, q^2 = \theta, v^1 = \dot{r}$ and $v^2 = \dot{\theta}$. If we need to determine the geometrical solution of the dynamics in the usual set of coordinates on TQ, we must pay attention to the fact that the term with $\partial/\partial q^i$ in the solution X in quasi-coordinates is given by (2). We have

$$X = w^m X_m + \left(F(q^1) + \frac{(w^2)^2}{(q^1)^3}\right)\frac{\partial}{\partial w^1} = w^m \beta^i_m \frac{\partial}{\partial q^i} + \left(F(q^1) + \frac{(w^2)^2}{(q^1)^3}\right)\frac{\partial}{\partial w^1}.$$

So, applying (2), the solution of the dynamics in the usual coordinates (q^1, q^2, v^1, v^2) on TQ is

$$X = v^i \left.\frac{\partial}{\partial q^i}\right|_{v\,\mathrm{const}} + v^i \frac{\partial \beta^j_k}{\partial q^i}\alpha^k_r v^r \frac{\partial}{\partial v^j} + \left(F(q^1) + \frac{(w^2)^2}{(q^1)^3}\right)\beta^i_1 \frac{\partial}{\partial v^i},$$

or equivalently,

$$X = v^i \frac{\partial}{\partial q^i}\bigg|_{v\,\mathrm{const}} + v^1 \frac{\partial}{\partial v^2}\left(\frac{-2}{r^3}\right) r^2 v^2 + \left(F(q^1) + \frac{(w^2)^2}{(q^1)^3}\right)\frac{\partial}{\partial v^1}.$$

Then, in the coordinates $(r, \theta, \dot r, \dot\theta)$ we have

$$X = \dot r \frac{\partial}{\partial r} + \dot\theta \frac{\partial}{\partial \theta} + (F(r) + r\dot\theta^2)\frac{\partial}{\partial \dot r} - \frac{2\dot r\dot\theta}{r}\frac{\partial}{\partial \dot\theta}.$$

Example 4.2. Let G be a Lie group and denote the neutral element of the group by e. We can identify the tangent bundle TG with $G \times T_eG$, using the map $TL_{g^{-1}} : TG \to G \times T_eG$ given by

$$TL_{g^{-1}}(g, \dot g) = (g, \xi),$$

where $L_g : G \to G$ is the left-translation defined by $L_g h = gh$, for all $h \in G$. If (ξ^I), $I = 1, \ldots, \dim G$, is the set of coordinates of $\xi \in T_eG$ with respect to a basis $\{e_I\}$ of T_eG, then we can define a set of quasi-velocities (ξ^I) on TG by

$$\xi^I e_I = \xi = T_g L_{g^{-1}}(\dot g).$$

If g is a point in G with local coordinates (g^I), then (g^I, ξ^I) defines a set of quasi-coordinates in TG. Note that the map $\alpha = T_g L_{g^{-1}} : T_gG \to T_eG$ is an invertible linear transformation whose inverse transformation is given by $\beta = T_e L_g : T_eG \to T_gG$. Thus, $\xi^I = \alpha^I_J \dot g^J$ and $\dot g^I = \beta^I_J \xi^J$, where $\alpha = (\alpha^I_J)$ and $\beta = (\beta^I_J)$ are the coordinate transformation matrices between the usual coordinates $(g^I, \dot g^I)$ on TG and the quasi-coordinates (g^I, ξ^I).

A (regular) G-invariant Lagrangian $\mathcal{L} \in C^\infty(TG)$ is given in quasi-coordinates by a function l in T_eG as follows: $\mathcal{L}(g, \dot g) = l(\xi)$. As the Lagrangian $\mathcal{L}$ on the Lie algebroid TG is G-invariant, it is then possible to obtain the Euler–Lagrange equations of the gauge algebroid $TG/G \equiv \mathfrak{g}$ from the equations (3) in TG:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial l}{\partial \xi^I}\right) = \xi^J c^K_{JI} \frac{\partial l}{\partial \xi^K}, \tag{8}$$

where c^K_{JI} are the structure constants of the Lie algebra $\mathfrak{g}$ of the Lie group G with respect to the basis $\{e_I\}$ of $\mathfrak{g} \equiv T_eG$. In fact, let $X^L(g) = T_e L_g(X)$ be the left-invariant vector field in G associated with an element X in the Lie algebra $\mathfrak{g}$ of G. Thus, the set $\{e^L_I\}$ represents a local basis of sections of TG associated with the set of local coordinates (g^I, ξ^I) on $TG \equiv G \times \mathfrak{g}$, because

$$\xi^I e^L_I = \dot g^i \frac{\partial}{\partial g^i}.$$

The structure functions of the Lie algebroid TG are given by

$$\left[e^L_I, e^L_J\right]_{TG} = c^K_{IJ} e^L_K, \qquad \rho_{TG}\left(e^L_J\right) = e^L_J.$$

Using the structure functions of the Lie algebroid TG and the Euler–Lagrangian equations (3), in the absence of non-conservative forces, we obtain (8). Then, the geometric solution of the dynamics is a SODE vector field on TG given by $X_{\mathcal{L}} = \xi^M e^L_M + f^M \partial/\partial \xi^M$, with

$$f^M = \overline{W}^{MI} \xi^J c^K_{JI} \frac{\partial l}{\partial \xi^K},$$

where $\overline{W}^{MI}$ is the inverse matrix of $(\partial^2 l/\partial \xi^I \partial \xi^M)$.

5. Changes of local coordinates on a Lie algebroid with fixed base coordinates

As in the geometric approach to Lagrangian formulation of classical mechanics, it is possible to write the dynamics on a Lie algebroid using different sets of coordinates [16, 19], as happens with the quasi-coordinates formalism on a tangent bundle [11, 14]. But in an arbitrary Lie algebroid we do not have a canonical basis of sections. Therefore, there is no natural set of coordinates for a Lie algebroid. Sometimes, however, it is worthwhile writing the equations of motion in a given set instead of another initially given, as we shall see in the following section.

Recall that the *prolongation of the Lie algebroid* $p : A \to Q$ (see [13, 16, 19]) is a vector bundle $\mathcal{T}A$ over A, where $\mathcal{T}A$ is the total space of the pullback of the vector bundle $Tp : TA \to TQ$ by the anchor map $\rho : A \to TQ$. The projection $p_{\mathcal{T}A} : \mathcal{T}A \to A$ is defined by $p_{\mathcal{T}A}(b, v) = p_{TA}(v) = a \in A$, with $p_{TA} : TA \to A$ being the canonical projection of the tangent bundle TA over the base A. An element (b, v) of $\mathcal{T}A$ will be denoted by (a, b, v), where $v \in T_aA$. With this notation, $\mathcal{T}A = \{(a, b, v) \in A \times A \times TA | p(a) = p(b), \rho(b) = T_a p(v), \text{ with } v \in T_aA\}$. The vector bundle $p_{\mathcal{T}A} : \mathcal{T}A \to A$ can be endowed with a Lie algebroid structure, where the anchor is the map $\rho_{\mathcal{T}A} : \mathcal{T}A \to TA$, given by $\rho_{\mathcal{T}A}(a, b, v) = v$, and the Lie bracket on the space of sections is defined by setting [16, 19]:

$$[V_1, V_2]_{\mathcal{T}A}(a) = (a, [\sigma_1, \sigma_2]_A(p(a)), [X_1, X_2](a)),$$

for all $a \in A$ and all projectable sections $V_1, V_2 \in \Gamma(\mathcal{T}A)$, i.e. sections of the form $V_i(a) = (a, \sigma_i(p(a)), X_i(a))$, where $\sigma_i \in \Gamma(A)$ and $X_i \in \mathfrak{X}(A)$ are such that $Tp \circ X_i = \rho(\sigma_i) \circ p$, with $i = 1, 2$. If A is the tangent bundle to a manifold Q, $A = TQ$, endowed with its usual Lie algebroid structure, the prolongation of the Lie algebroid A is the tangent bundle $T(TQ)$ to TQ endowed with its usual structure of Lie algebroid over TQ [19].

Let us consider on the Lie algebroid A a new set of local coordinates $\{(q^i, \mathbf{w}^\alpha) | i = 1, \ldots, n, \alpha = 1, \ldots, s\}$ (see section 3) associated with a basis of sections $\{f_\alpha | \alpha = 1, \ldots, s\}$ of A, that satisfies

$$\mathbf{w}^\alpha = \widehat{\Phi_\alpha}(q, \mathbf{v}) = \Phi_{\alpha\beta}(q)\mathbf{v}^\beta, \qquad \mathbf{v}^\alpha = \widehat{\Psi_\alpha}(q, \mathbf{w}) = \Psi_{\alpha\beta}(q)\mathbf{w}^\beta, \tag{9}$$

for all $\alpha = 1, \ldots, s$, where $\widehat{\Phi_\alpha}$ and $\widehat{\Psi_\alpha}$ are linear functions on A associated with the A-1-forms Φ_α and Ψ_α, respectively, such that $\Psi_{\alpha\beta}\Phi_{\beta\gamma} = \delta_{\alpha\gamma}$. Associated with the new coordinates on the Lie algebroid A, we consider on the prolongation of A the following basis of local sections:

$$\mathcal{X}'_\alpha(a) = (a, f_\alpha(p(a)), X_\alpha(a)), \qquad \mathcal{V}'_\alpha(a) = \left(a, 0, \left.\frac{\partial}{\partial \mathbf{w}^\alpha}\right|_a\right), \tag{10}$$

where $X_\alpha = \Psi_{\beta\alpha}\rho^i_\beta \partial/\partial q^i$, for all $\alpha = 1, \ldots, r$. Thus, the structure functions of the Lie algebroid $\mathcal{T}A$ are given by

$$[\mathcal{X}'_\alpha, \mathcal{X}'_\beta]_{\mathcal{T}A} = \gamma^\epsilon_{\alpha\beta}\mathcal{X}'_\epsilon, \qquad [\mathcal{X}'_\alpha, \mathcal{V}'_\beta]_{\mathcal{T}A} = 0, \qquad [\mathcal{V}'_\alpha, \mathcal{V}'_\beta]_{\mathcal{T}A} = 0,$$

$$\rho_{\mathcal{T}A}(\mathcal{X}'_\alpha) = X_\alpha, \qquad \rho_{\mathcal{T}A}(\mathcal{V}'_\alpha) = \frac{\partial}{\partial \mathbf{w}^\alpha},$$

where $[f_\alpha, f_\beta]_A = \gamma^\epsilon_{\alpha\beta} f_\epsilon$.

Let $\mathcal{L} \in C^\infty(A)$ be the Lagrangian of a dynamical system on the Lie algebroid A, subject to the action of a non-conservative force $\mathcal{Q}$. If the Lagrangian is regular, the dynamics has a unique solution $X = a^\alpha \mathcal{X}'_\alpha + b^\alpha \mathcal{V}'_\alpha$ that satisfies the system (see [4])

$$\begin{cases} a^\alpha = \mathbf{w}^\alpha \\ b^\alpha = \overline{W}^{\alpha\beta}\left[\mathbf{w}^\zeta \gamma^\epsilon_{\zeta\beta}\dfrac{\partial \mathcal{L}}{\partial \mathbf{w}^\epsilon} - \mathbf{w}^\zeta X_\zeta\left(\dfrac{\partial \mathcal{L}}{\partial \mathbf{w}^\beta}\right) + X_\beta(\mathcal{L}) + \Upsilon_\beta\right], \end{cases}$$

where $\overline{W}^{\alpha\beta}$ represent the entries of the inverse matrix of $\left(\partial^2\mathcal{L}/\partial\mathbf{w}^\beta\partial\mathbf{w}^\alpha\right)$ and $\Upsilon_\beta = \Psi_{\alpha\beta}Q_\alpha$ is the β-component of the non-conservative force $\mathcal{Q} = Q_\alpha\mathcal{X}^\alpha$, in the new coordinates. The solution of the dynamics is a SODE section of $\mathcal{T}A$ because $S(X) = \Delta$, and the dynamical equation is equivalent to

$$\pounds_{\rho_{\mathcal{T}A}}(X)\theta_{\mathcal{L}} = d_{\mathcal{T}A}\mathcal{L} + \mathcal{Q},$$

where $\pounds_{\rho_{\mathcal{T}A}}(X) := \mathrm{i}(X)\circ d_{\mathcal{T}A} + d_{\mathcal{T}A}\circ \mathrm{i}(X)$. The generalized Euler–Lagrange equations in the new coordinates are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\mathbf{w}^\alpha}\right) = \Psi_{\beta\alpha}\rho^i_\beta\frac{\partial\mathcal{L}}{\partial q^i} + \mathbf{w}^\epsilon\gamma^\beta_{\epsilon\alpha}\frac{\partial\mathcal{L}}{\partial\mathbf{w}^\beta} + \Upsilon_\alpha, \tag{11}$$

where $\dot q^i = \mathbf{w}^\alpha\Psi_{\beta\alpha}\rho^i_\beta$. The above equations are the Lagrange equations given by Weinstein [28] in the new set of coordinates $(q^i, \mathbf{w}^\alpha)$ that corresponds to the local basis $\{f_\alpha|\alpha = 1,\ldots,s\}$ of sections of A.

Example 5.1. Let $P(M,G)$ be a principal bundle and $(q^i, g^I, \dot q^i, \xi^I)$ be a set of quasi-coordinates on TP, where (q^i) are local coordinates on M, $(q^i, \dot q^i)$ are the usual local coordinates on the tangent bundle TM and (g^I, ξ^I) is a set of quasi-coordinates on TG, as given in example 4.2.

Let $\mathcal{L} \in C^\infty(TP)$ be a regular G-invariant Lagrangian. It is given in terms of quasi-coordinates by $\mathcal{L}_{qc}(q, g, \dot q, \xi) = \mathcal{L}(q, g, \dot q, \dot g) = l(q, \dot q, \xi)$, and we can write the equations of motion on the gauge algebroid TP/G from equations (11), in the absence of non-conservative forces, as follows (see [14]):

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial l}{\partial\xi^I}\right) = \xi^J\gamma^K_{JI}\frac{\partial l}{\partial\xi^K}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial l}{\partial\dot q^i}\right) = \frac{\partial l}{\partial q^i}, \tag{12}$$

where $\gamma^K_{JI} = c^K_{JI}$ are the structure constants of the Lie algebra $\mathfrak{g}$ of the Lie group G, with respect to a basis $\{e_I\}$ of $\mathfrak{g} \equiv T_eG$. Indeed, let $\Pi : TP \to TP/G$ be the canonical projection over the principal bundle $\pi : P \to M = P/G$. The gauge algebroid structure $(\rho_{TP/G}, [\cdot,\cdot]_{TP/G})$ is given by (see for example [3]):

(1) $\rho_{TP/G}(\widehat{X}) = T\pi \circ \rho_{TP}(X)$,

(2) $[\widehat{X}, \widehat{Y}]_{TP/G} = \Pi \circ [X, Y]_{TP}$,

for all $X, Y \in \Gamma(TP)$, Π-related with $\widehat{X}, \widehat{Y} \in \Gamma(TP/G)$, respectively. Let us consider a connection on the principal bundle $\pi : P \to M$ and denote by $\mathcal{A}$ the associated connection 1-form; in local coordinates, $\mathcal{A}(\partial/\partial q^i|_q) = A^I_i(q)e_I$. Let $\{e_i, e_I\}$ be a basis of local sections of TP/G, obtained from the Π-projection of the basis $\left\{(\partial/\partial q^i)^h, e^L_I\right\}$ of local sections of TP, where $(\partial/\partial q^i)^h = \partial/\partial q^i - A^I_i e^L_I$ denotes the horizontal lift to P of the vector field $\partial/\partial q^i$ on M and e^L_I is the left-invariant vector field on G corresponding to the element e_I of the basis of $\mathfrak{g}$. In these local coordinates, the gauge algebroid structure is given by

$$[e_i, e_j]_{TP/G} = -C^K_{ij}e_K \qquad [e_i, e_I]_{TP/G} = c^K_{IJ}A^J_i e_K, \qquad [e_I, e_J]_{TP/G} = c^K_{IJ}e_K,$$

$$\rho_{TP/G}(e_i) = \frac{\partial}{\partial q^i}, \qquad \rho_{TP/G}(e_J) = 0,$$

where C^K_{ij} are the coefficients of the curvature form of the chosen principal connection. The local basis of sections $\left\{\partial/\partial q^i, e^L_I\right\}$ of TP is the one associated with the set of local coordinates $(q^i, g^I, \dot q^i, \xi^I)$ initially given. The Π-projections, $\Pi\circ\partial/\partial q^i = f_i\circ\pi$ and $\Pi\circ e^L_I = f_I\circ\pi$, define a local basis of sections $\{f_i, f_I\}$ of TP/G associated with the local coordinates $(q^i, \dot q^i, \xi^I)$ on TP/G. Note that $f_i = e_i + A^I_i e_I$ and $f_I = e_I$. According to the definition of the gauge algebroid structure, we have $[f_i, f_j]_{TP/G} = [f_i, f_I]_{TP/G} = 0$ and $[f_I, f_J]_{TP/G} = c^K_{IJ} f_K$. So, using the reduced Lagrangian $l \in C^\infty(TP/G)$, we obtain the equations of motion (12) from the generalized Euler–Lagrange equations (11). The reduced dynamics $\widehat{X}_l = a^\alpha \mathcal{X}'_\alpha + b^\alpha \mathcal{V}'_\alpha$ satisfies the following system

$$\begin{cases} a^\alpha = \mathbf{w}^\alpha \\ b^\alpha = \overline{W}^{\alpha\beta}\left[\xi^J \gamma^K_{J\beta}\dfrac{\partial l}{\partial \xi^K} - \dot{q}^i \dfrac{\partial^2 l}{\partial q^i \partial \mathbf{w}^\beta} + \delta_{\beta i}\dfrac{\partial l}{\partial q^i}\right], \end{cases}$$

where $\mathbf{w}^i = \dot{q}^i, \mathbf{w}^I = \xi^I$, $\overline{W}^{\alpha\beta}$ are the entries of the inverse matrix of $(\partial^2\mathcal{L}/\partial\mathbf{w}^\beta\partial\mathbf{w}^\alpha)$ and $\{\mathcal{X}'_\alpha, \mathcal{V}'_\alpha\}$ is the basis of local sections of $\mathcal{T}(TP/G)$ defined in (10).

6. Lagrangian systems with non-holonomic linear constraints

In this section we will study systems with linear non-holonomic constraints on a Lie algebroid A, i.e. constraints which are linear in the local coordinates $\mathbf{v}^\alpha$ on A associated with a local basis of sections $\{e_\alpha\}$ of A, by choosing local coordinates that are adapted to the constraints.

Consider a system with k linear non-holonomic constraints on a Lie algebroid $(A, \rho, [\cdot,\cdot]_A)$ over Q,

$$\phi_a(q, \mathbf{v}) = \widehat{\Phi_a}(q, \mathbf{v}) = \Phi_{a\beta}(q)\mathbf{v}^\beta,$$

given by a sub-bundle $\tau : B \to Q$ of A, where Φ_a denotes an A-1-form and $\widehat{\Phi_a}$ is the associated linear function. The submanifold B is defined by the set $\{\phi_a = 0 | a = 1, \ldots, k\}$ and is called the *constrained manifold*. Suppose that the A-1-forms Φ_a are such that $\Phi_1 \wedge \cdots \wedge \Phi_k \neq 0$. Then, the functions ϕ_a are functionally independent.

Let $\mathcal{T}B$ be the vector bundle over B, whose total space

$$\mathcal{T}B = \{(b, c, v) \in B \times B \times TB | \tau(b) = \tau(c), \varrho(c) = T\tau(v) \text{ with } v \in T_bB\}$$

is given by the pullback of the vector bundle $T\tau : TB \to TQ$ by the map $\varrho = \rho \circ \iota : B \to TQ$, where $\iota : B \to A$ is the canonical inclusion. The projection $p_{\mathcal{T}B} : \mathcal{T}B \to B$ of $\mathcal{T}B$ onto B, is given by $p_{\mathcal{T}B}(b, c, v) = b$. Suppose that $\mathcal{L} \in C^\infty(A)$ is a regular Lagrangian describing the non-holonomic system under the action of a non-conservative force $\mathcal{Q}$. Alike the formalism of linear non-holonomic systems in a tangent bundle, the equations of motion of the non-holonomic system in A can be written according to d'Alembert–Chetaev principle [18, 26], in the global form

$$\begin{cases} (\mathrm{i}(X)\omega_{\mathcal{L}} - d_{\mathcal{T}A}E_{\mathcal{L}} + \mathcal{Q})\,|_B \in \Gamma(\widetilde{B^0}) \\ X|_B \in \Gamma(\mathcal{T}B), \end{cases} \tag{13}$$

where $B^0 = \langle \Phi_a | a = 1, \ldots, k\rangle$ is the annihilator of B and $\widetilde{B^0} = \langle p_2^*\Phi_a | a = 1, \ldots, k\rangle$ can be viewed as a vector bundle over B, where $p_2 : \mathcal{T}A \to A$ is the projection defined by $p_2(a, b, v) = b$, for all $(a, b, v) \in \mathcal{T}A$. Note that for each $q \in Q$, B^0_q is the set generated by the elements $\Phi_a(q)$ satisfying $\langle\Phi_a(q), v\rangle = 0$, for each element $v \in B_q$ and $a = 1, \ldots, k$. Moreover, $\dim(\mathcal{T}^AB)^0 = k$, $(\mathcal{T}^AB)^0$ is generated by the 1-forms $d_{\mathcal{T}A}\phi_a$ and $S^*((\mathcal{T}^AB)^0) = \widetilde{B^0}$.

6.1. Lagrange multipliers method in a Lie algebroid framework

The non-holonomic system previously given can be studied using the method of Lagrangian multipliers (see [2] for the classical case). The dynamics equation of the system is given by

$$\mathrm{i}(X)\omega_{\mathcal{L}} = d_{\mathcal{T}A}E_{\mathcal{L}} - \mathcal{Q} - \lambda_a p_2^*\Phi_a, \tag{14}$$

where the Lagrange multipliers $\lambda_a \in C^\infty(A)$ are determined by the *tangency condition* $\pounds_{\rho_{\mathcal{T}A}}(X)\phi_a = 0$, for all $a = 1, \ldots, k$. Recall that $p_2 : \mathcal{T}A \to A$ is defined by $p_2(a, b, v) = b$ and the semi-basic sections $p_2^*\Phi_a = \Phi_{a\beta}\chi^\beta$ are the *reaction forces* of the Lie algebroid A (see [5]). The solution of equation (14) is a section of the form

$$X = X_{\mathcal{L}}^{\mathcal{Q}} + \lambda_a Z_a,$$

where $X_{\mathcal{L}}^{\mathcal{Q}}$ is the solution of the free dynamics (without constraints) and Z_a is the vertical section of $\mathcal{T}A$ such that

$$\mathrm{i}(Z_a)\omega_{\mathcal{L}} = -p_2^*\Phi_a.$$

Note, once again, that X is a SODE section of $\mathcal{T}A$, since $S(X) = S\big(X_{\mathcal{L}}^{\mathcal{Q}}\big) = \Delta$.

It is important to observe that for computing the Lagrangian multipliers λ_a, we must suppose the following *compatibility condition* [7]: the matrix of entries $\mathcal{C}_{ab} = \rho_{\mathcal{T}A}(Z_a)\phi_b$ is regular in each point of B, where $B = \{\mathbf{w}^\alpha = 0 | \alpha = s - k + 1, \ldots, s\}$ is the constrained manifold. In this case, we say that the non-holonomic system $(\mathcal{L}, B)$ on the Lie algebroid A is *regular*, what is assumed hereafter. We can also assume without losing generality that the last k columns of the matrix $(\Phi_{a\beta})$ are independent.

Under the above assumptions, consider a set of coordinates $(q^1, \ldots, q^n, \mathbf{w}^1, \ldots, \mathbf{w}^s)$ adapted to the constraints on the bundle A:

$$\mathbf{w}^\alpha = \mathbf{v}^\alpha, \qquad \forall\, \alpha = 1, \ldots, (s - k),$$
$$\mathbf{w}^{s-k+a} = \phi_a, \qquad \forall\, a = 1, \ldots, k.$$

The transformation matrices are

$$\widetilde{\Phi} = \begin{pmatrix} I_{s-k} & 0_{(s-k)\times k} \\ C_{21} & C_{22} \end{pmatrix} \qquad \text{and} \qquad \widetilde{\Psi} = \begin{pmatrix} I_{s-k} & 0_{(s-k)\times k} \\ D_{21} & D_{22} \end{pmatrix},$$

where $C = (C_{21} C_{22})$ is given by $C_{ab} = \Phi_{ab}$, for all $a = 1, \ldots, k$ and $b = 1, \ldots, s$, C_{22} is invertible by hypothesis and the matrix $D = (D_{21} D_{22})$ is given by $D_{21} = -C_{22}^{-1}C_{21}$ and $D_{22} = C_{22}^{-1}$. These matrices satisfy $\widetilde{\Phi}\widetilde{\Psi} = I_s = \widetilde{\Psi}\widetilde{\Phi}$.

The geometrical dynamics solution is given in the new coordinates by

$$X|_B = \mathbf{w}^{\underline{\alpha}}\mathcal{X}'_{\underline{\alpha}} + f^{\underline{\alpha}}(q, \mathbf{w})\mathcal{V}'_{\underline{\alpha}},$$

where $\underline{\alpha} = 1, \ldots, s - k$. The functions $f^{\underline{\alpha}}$ are determined by

$$f^{\underline{\alpha}}(q, \mathbf{w}) = g^{\underline{\alpha}}(q, \mathbf{w}) + \lambda_a \mathcal{W}^{\underline{\alpha}\beta}\Phi_{a\beta},$$

with $\beta = 1, \ldots, s$ and $a = 1, \ldots, k$, where

$$X_{\mathcal{L}}^{\mathcal{Q}} = \mathbf{w}^\beta \mathcal{X}'_\beta + g^\beta(q, \mathbf{w})\mathcal{V}'_\beta, \qquad Z_a = \mathcal{W}^{\alpha\beta}\Phi_{a\beta}\mathcal{V}'_\alpha,$$

and the function λ_a is given by

$$d_{\mathcal{T}A}\mathbf{w}^{n-k+b}\big(X_{\mathcal{L}}^{\mathcal{Q}}\big) + \lambda_a d_{\mathcal{T}A}\mathbf{w}^{n-k+b}(Z_a) = 0, \qquad \forall\, b = 1, \ldots, k.$$

Therefore, the dynamics is given by the integral curves of the following vector field in B:

$$\rho_{\mathcal{T}A}(X|_B) = \mathbf{w}^{\underline{\alpha}} X_{\underline{\alpha}} + f^{\underline{\alpha}}\frac{\partial}{\partial \mathbf{w}^{\underline{\alpha}}},$$

and satisfies

$$\pounds_{\rho_{\mathcal{T}A}}(X)\theta_{\mathcal{L}} = d_{\mathcal{T}A}\mathcal{L} + \mathcal{Q} + \lambda_a p_2^*\Phi_a$$

because X is a SODE section of $\mathcal{T}A$. In the new coordinates, the previous equation is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \mathbf{w}^\alpha}\right) = \widetilde{\Psi}_{\beta\alpha}\rho^i_\beta \frac{\partial \mathcal{L}}{\partial q^i} + \mathbf{w}^{\underline{\epsilon}}\gamma^\beta_{\underline{\epsilon}\alpha}\frac{\partial \mathcal{L}}{\partial \mathbf{w}^\beta} + \Upsilon_\alpha + \lambda_a \widetilde{\Psi}_{\beta\alpha}\Phi_{a\beta},$$

where $\dot{q}^i = \mathbf{w}^{\underline{\alpha}}\widetilde{\Psi}_{\beta\underline{\alpha}}\rho^i_\beta$ and $\Upsilon_\alpha = \widetilde{\Psi}_{\beta\alpha}\mathcal{Q}_\beta$ is the α-component of the non-conservative force $\mathcal{Q}$ on the new coordinates, with $\underline{\alpha}, \underline{\epsilon} = 1, \ldots, s-k$; $\alpha, \beta = 1, \ldots, s$; $a = 1, \ldots, k$ and $i = 1, \ldots, n$. Thus, we have

$$\begin{cases} \dot{q}^i = \mathbf{w}^{\underline{\alpha}}\widetilde{\Psi}_{\beta\underline{\alpha}}\rho^i_\beta \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial\mathcal{L}}{\partial\mathbf{w}^{\underline{\alpha}}}\right) = \widetilde{\Psi}_{\beta\underline{\alpha}}\rho^i_\beta\dfrac{\partial\mathcal{L}}{\partial q^i} + \mathbf{w}^{\underline{\epsilon}}\gamma^\beta_{\underline{\epsilon\alpha}}\dfrac{\partial\mathcal{L}}{\partial\mathbf{w}^\beta} + \Upsilon_{\underline{\alpha}} \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial\mathcal{L}}{\partial\mathbf{w}^{\bar{\alpha}}}\right) = \widetilde{\Psi}_{\beta\bar{\alpha}}\rho^i_\beta\dfrac{\partial\mathcal{L}}{\partial q^i} + \mathbf{w}^{\underline{\epsilon}}\gamma^\beta_{\underline{\epsilon}\bar{\alpha}}\dfrac{\partial\mathcal{L}}{\partial\mathbf{w}^\beta} + \Upsilon_{\bar{\alpha}} + \lambda_{\bar{\alpha}-s+k}, \end{cases}$$

with $\bar{\alpha} = s-k+1, \ldots, s$. In this system, after the elimination of the Lagrange multipliers we obtain, in the new coordinates, the generalized Lagrange–d'Alembert equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\mathbf{w}^{\underline{\alpha}}}\right) = \widetilde{\Psi}_{\beta\underline{\alpha}}\rho^i_\beta\frac{\partial\mathcal{L}}{\partial q^i} + \mathbf{w}^{\underline{\epsilon}}\gamma^\beta_{\underline{\epsilon\alpha}}\frac{\partial\mathcal{L}}{\partial\mathbf{w}^\beta} + \Upsilon_{\underline{\alpha}},$$

where $\dot{q}^i = \mathbf{w}^{\underline{\alpha}}\widetilde{\Psi}_{\beta\underline{\alpha}}\rho^i_\beta$.

When selecting a different set of coordinates adapted to the constraints, similar results would be obtained: $\mathbf{w}^I = \widetilde{\Phi}_{I\beta}\mathbf{v}^\beta$ and $\mathbf{w}^{s-k+a} = \phi_a = \Phi_{a\beta}\mathbf{v}^\beta$, for all $I = 1, \ldots, s-k$ and $a = 1, \ldots, k$. The only condition that has to be assumed is the invertibility of the matrix

$$\widetilde{\Phi} = \begin{pmatrix} C \\ D \end{pmatrix},$$

where $C_{I\beta} = \widetilde{\Phi}_{I\beta}$ and $D_{a\beta} = \Phi_{a\beta}$, for $I = 1, \ldots, s-k$; $a = 1, \ldots, k$ and $\beta = 1, \ldots, s$.

Example 6.1. Consider the motion of a free particle of unity mass in the configuration space $M = \mathbb{R}^3$, with a linear constraint

$$\phi = \dot{z} - y\dot{x}.$$

In order to determine the solution of this problem, we consider the set of local coordinates $(x, y, z, \mathbf{w}^1, \mathbf{w}^2, \mathbf{w}^3)$, on the Lie algebroid $A = T\mathbb{R}^3$ given by

$$\mathbf{w}^1 = v_x = \dot{x}, \qquad \mathbf{w}^2 = v_y = \dot{y}, \qquad \mathbf{w}^3 = \phi = v_z - yv_x,$$

whose coordinates transformation matrices $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are

$$\widetilde{\Phi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -y & 0 & 1 \end{pmatrix}, \qquad \widetilde{\Psi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ y & 0 & 1 \end{pmatrix}.$$

The motion of the free particle is characterized by the regular Lagrangian

$$\mathcal{L} = \tfrac{1}{2}((\mathbf{w}^1)^2 + (\mathbf{w}^2)^2 + (\mathbf{w}^3 + y\mathbf{w}^1)^2).$$

Let B be the constraint manifold. When solving the problem through the method of Lagrangian multipliers, we obtain the solution

$$X|_B = \mathbf{w}^1\mathcal{X}'_1 + \mathbf{w}^2\mathcal{X}'_2 - \frac{\mathbf{w}^1\mathbf{w}^2}{y^2+1}y\mathcal{V}'_1,$$

where

$$X_{\mathcal{L}} = \mathbf{w}^1\mathcal{X}'_1 + \mathbf{w}^2\mathcal{X}'_2 + \mathbf{w}^3\mathcal{X}'_3 - \mathbf{w}^1\mathbf{w}^2\mathcal{V}'_3,$$
$$Z = -y\mathcal{V}'_1 + (y^2+1)\mathcal{V}'_3,$$
$$\lambda = \frac{\mathbf{w}^1\mathbf{w}^2}{y^2+1}.$$

In this particular case, we can identify $\mathcal{T}A = \mathcal{T}(T\mathbb{R}^3)$ with $T(T\mathbb{R}^3) = TA$. In a similar way, we can identify the solution $X|_B$ with a vector field on B,

$$X|_B \equiv \mathbf{w}^1 \frac{\partial}{\partial x} + \mathbf{w}^2 \frac{\partial}{\partial y} + y\mathbf{w}^1 \frac{\partial}{\partial z} - \frac{\mathbf{w}^1\mathbf{w}^2}{y^2+1} y \frac{\partial}{\partial \mathbf{w}^1},$$

and the dynamics is then given by the integral curves of this vector field.

6.2. Gibbs–Appell's method in a Lie algebroid framework

In the Lie algebroid formalism, the aim of Gibbs–Appell's method is to determine the equations of motion of a system with constraints [11, 12]. This method consists, in a first step, on determining Gibbs–Appell's function associated with the Lagrangian of the system without constraints, and then to express this function in terms of a set of coordinates adapted to the constraints. In a final step, we need to determine the equations of motion given by Gibbs–Appell's method in the new coordinates.

Next we will determine Gibbs–Appell's function associated with a Lagrangian $\mathcal{L} \in C^\infty(A)$, defined on a Lie algebroid $(A, \rho, [\cdot,\cdot]_A)$ over Q. We will show that this function is defined on a subset $A^{(2)}$ of TA, given by the set of equivalent classes of *admissible curves* in the bundle $p : A \to Q$, i.e. curves in $A, \sigma' : I \to A$, such that $\dot{\sigma}(t) = \rho(\sigma'(t))$, where $\sigma : I \to Q$ is a curve in Q given by $\sigma = p \circ \sigma'$. Two curves in $A, \sigma_A \equiv (\sigma, \sigma')$ and $\gamma_A \equiv (\gamma, \gamma')$ are said to be equivalent if

$$\begin{cases} \sigma(0) = \gamma(0) \\ \sigma'(0) = \gamma'(0) \\ \dot{\sigma}'(0) = \dot{\gamma}'(0). \end{cases}$$

Equivalently, $A^{(2)}$ can be defined as $A^{(2)} = \bigcup_{a\in A} A_a^{(2)}$, with

$$A_a^{(2)} = \{v \in T_aA | T_a p(v) = \rho(a)\}.$$

The set $A^{(2)}$ is an affine sub-bundle of $p_{TA} : TA \to A$, modelled over the fibre bundle $\mathrm{Ker}(Tp)$, whose projection on A is given by

$$p_{2,1} : (q, \mathbf{v}, \mathbf{a}) \in A^{(2)} \to (q, \mathbf{v}) \in A.$$

Moreover, the inclusion of $A^{(2)}$ in TA is defined by

$$i : (q^i, \mathbf{v}^\alpha, \mathbf{a}^\alpha) \in A^{(2)} \to \left(q^i, \mathbf{v}^\alpha, \mathbf{v}^\alpha\rho^i_\alpha, \mathbf{a}^\alpha\right) \in TA.$$

So, given an element $v \in A_a^{(2)}$ of the form $v = (q^i, \mathbf{v}^\alpha, \mathbf{a}^\alpha)$, we have $\mathrm{i}(v) = \mathbf{v}^\alpha \rho^i_\alpha \partial/\partial q^i + \mathbf{a}^\alpha \partial/\partial \mathbf{v}^\alpha$.

Let $\mathcal{L} \in C^\infty(A)$ be a regular Lagrangian of a system without constraints and $\mathcal{Q}$ a non-conservative force. To define Gibbs–Appell's function associated with the Lagrangian $\mathcal{L}$, we need to consider the section $\Gamma_\mathcal{L}$ of $\mathcal{T}A$ along the map $p_{2,1} : A^{(2)} \to A$,

$$\Gamma_\mathcal{L} = X_\mathcal{L} \circ p_{2,1} - \mathbf{T}^{(1)},$$

where $X_\mathcal{L}$ is the solution of the equation $\mathrm{i}(X_\mathcal{L})\omega_\mathcal{L} = d_{\mathcal{T}A}E_\mathcal{L}$, given in local coordinates by $X_\mathcal{L}(q, \mathbf{v}) = \mathbf{v}^\alpha \mathcal{X}_\alpha + F^\alpha(q, \mathbf{v})\mathcal{V}_\alpha$, and $\mathbf{T}^{(1)}$ is a section of $\mathcal{T}A$ along the map $p_{2,1} : A^{(2)} \to A$ defined by $v^{(1)} = \rho_{\mathcal{T}A} \circ \mathbf{T}^{(1)}$, where $v^{(1)}$ is a vector field on A along the map $p_{2,1} : A^{(2)} \to A$ given by

$$v^{(1)} \circ \sigma^2 = T\sigma^1(\mathrm{d}/\mathrm{d}t),$$

for all admissible curves $\sigma^1 \equiv (\sigma, \sigma') : I \to A$, where $\sigma^2 := (\sigma, \sigma', \dot{\sigma}') : I \to A^{(2)}$.

$$\begin{array}{ccc} \mathcal{T}A & \xrightarrow{\rho_{\mathcal{T}A}} & TA \\ \mathbf{T}^{(1)} \uparrow & \nearrow^{v^{(1)}} & \downarrow p_{TA} \\ A^{(2)} & \xrightarrow[p_{2,1}]{} & A \end{array}$$

In local coordinates, the curve σ^1 is given by $(\sigma^i, \mathbf{v}^\alpha)$ and, since it is admissible, we have $\dot{\sigma}^i = \mathbf{v}^\alpha \rho^i_\alpha$. Thus,

$$v^{(1)}(q^i, \mathbf{v}^\alpha, \dot{\mathbf{v}}^\alpha) = \mathbf{v}^\alpha \rho^i_\alpha(q)\partial_{q^i} + \dot{\mathbf{v}}^\alpha \frac{\partial}{\partial v^\alpha}.$$

The section $\mathbf{T}^{(1)}$ of $\mathcal{T}A$ along the map $p_{2,1} : A^{(2)} \to A$ is given, in local coordinates, by

$$\mathbf{T}^{(1)}(q^i, \mathbf{v}^\alpha, \dot{\mathbf{v}}^\alpha) = \mathbf{v}^\alpha \mathcal{X}_\alpha(q^i, \mathbf{v}^\alpha) + \dot{\mathbf{v}}^\alpha \mathcal{V}_\alpha(q^i, \mathbf{v}^\alpha).$$

Therefore, the section $\Gamma_\mathcal{L}$ takes values in the vertical sub-bundle of $\mathcal{T}A$, i.e. $p_2 \circ \Gamma_\mathcal{L} = s_0 \circ p_{2,1}$, where s_0 is the null section of A; in local coordinates,

$$\Gamma_\mathcal{L}(q^i, \mathbf{v}^\alpha, \dot{\mathbf{v}}^\alpha) = (F^\alpha(q^i, \mathbf{v}^\alpha) - \dot{\mathbf{v}}^\alpha)\mathcal{V}_\alpha.$$

Consider a symmetric tensor $\mathcal{G}_A : \mathcal{T}A \times_A \mathcal{T}A \to \mathbb{R}$ on A, given in local coordinates by

$$\mathcal{G}_A(q, \mathbf{v}) = \mathcal{G}^1_{\alpha\beta}(q, \mathbf{v})\mathcal{V}^\alpha \otimes \mathcal{V}^\beta + \mathcal{G}^2_{\alpha\beta}(q, \mathbf{v})\mathcal{V}^\alpha \otimes \mathcal{X}^\beta + \mathcal{G}^3_{\alpha\beta}(q, \mathbf{v})\mathcal{X}^\alpha \otimes \mathcal{X}^\beta,$$

such that $S^*\mathcal{G}_A = p_2^*\mathcal{G}$, where $\mathcal{G}$ is the *fundamental tensor* associated with the Lagrangian function $\mathcal{L} \in C^\infty(A)$. $\mathcal{G}$ is given, in local coordinates, by $\mathcal{G}(q, \mathbf{v}) = \mathcal{G}_{\alpha\beta}(q)e^\alpha \otimes e^\beta$, with $\mathcal{G}_{\alpha\beta} = \partial^2\mathcal{L}/\partial\mathbf{v}^\alpha\partial\mathbf{v}^\beta$. Note that the fundamental tensor $\mathcal{G} : A \to S^2A^*$ associated with the Lagrangian $\mathcal{L}$ is a bundle map over Q, from the Lie algebroid $p : A \to Q$ to the bundle $p_{S^2A^*} : S^2A^* = \bigcup_{q\in Q} S^2_qA^* \to Q$, where

$$S^2_qA^* = \{\mathcal{G}_q : A_q \times A_q \to \mathbb{R} | \mathcal{G}_q \text{is bilinear and symmetric}\}.$$

Consequently, $\mathcal{G}^1_{\alpha\beta}(a) = \mathcal{G}_{\alpha\beta}(p(a))$, for all $a \in A$. Gibbs–Appell's function associated with the Lagrangian $\mathcal{L}$ is a function on $A^{(2)}$ [17], defined by

$$G_\mathcal{L} = \tfrac{1}{2}\widetilde{\mathcal{G}}(\Gamma_\mathcal{L}, \Gamma_\mathcal{L}),$$

where $\widetilde{\mathcal{G}} = \mathcal{G}_A \circ p_{2,1}$.

Consider a dynamical system on the Lie algebroid A with k linear non-holonomic constraints $\phi_a(q, \mathbf{v}) = \widehat{\Phi_a}(q, \mathbf{v}) = \Phi_{a\beta}(q)\mathbf{v}^\beta$, and let $(q^1, \ldots, q^n, \mathbf{w}^1, \ldots, \mathbf{w}^s)$ be a set of coordinates on A adapted to the constraints

$$\mathbf{w}^\alpha = \widetilde{\Phi}_{\alpha\beta}\mathbf{v}^\beta,$$

where the last k coordinates coincide with the constraints ϕ_a, i.e.

$$\mathbf{w}^I = \widetilde{\Phi}_{I\beta}\mathbf{v}^\beta, \qquad \forall\, I = 1, \ldots, (s-k),$$
$$\mathbf{w}^{s-k+a} = \phi_a, \qquad \forall\, a = 1, \ldots, k.$$

The transformation matrix $\widetilde{\Phi}$ is invertible, so $\mathbf{v}^\alpha = \widetilde{\Psi}_{\alpha\beta}\mathbf{w}^\beta$, where $\widetilde{\Phi}_{\gamma\alpha}\widetilde{\Psi}_{\alpha\beta} = \delta_{\gamma\beta}$. The Gibbs–Appell function $G_\mathcal{L}$, defined on a curve $(q^i, \mathbf{v}^\alpha, \dot{\mathbf{v}}^\alpha)$ in $A^{(2)}$, is given by

$$G_\mathcal{L}(q, \mathbf{v}, \dot{\mathbf{v}}) = \tfrac{1}{2}\mathcal{G}_{\alpha\beta}(q)\dot{\mathbf{v}}^\alpha\dot{\mathbf{v}}^\beta - \mathcal{G}_{\alpha\beta}(q)\dot{\mathbf{v}}^\alpha F^\beta(q, \mathbf{v}) + \tfrac{1}{2}\mathcal{G}_{\alpha\beta}(q)F^\alpha(q, \mathbf{v})F^\beta(q, \mathbf{v}).$$

In the case where both the constraints $\mathbf{w}^{s-k+a} = \phi_a$ and their time derivatives are equal to zero, the function $G_\mathcal{L}$ is easily written in the new coordinates $(q^1, \ldots, q^n, \mathbf{w}^1, \ldots, \mathbf{w}^{s-k}, \dot{\mathbf{w}}^1, \ldots, \dot{\mathbf{w}}^{s-k})$. Note that the time derivative of a function $f \in C^\infty(A)$ is a function in $C^\infty(A^{(2)})$, given by

$$d_{\mathbf{T}^{(1)}} f = i(\mathbf{T}^{(1)})d_{\mathcal{T}A} f = v^{(1)} f,$$

where $\mathbf{T}^{(1)}$ is the section of $\mathcal{T}A$ over the map $p_{2,1} : A^{(2)} \to A$ defined above.

In order to determine the equations of motion of the non-holonomic system depending on a non-conservative force $\mathcal{Q}$, we need to solve the system given by the Gibbs–Appell $(s-k)$ equations

$$\frac{\partial G_{\mathcal{L}}(q, \mathbf{w}, \dot{\mathbf{w}})}{\partial \dot{\mathbf{w}}^I} = \Upsilon_I, \tag{15}$$

where $\Upsilon_I = \widetilde{\Psi}_{\beta I} Q_\beta$ is the I-component of the non-conservative force, in the new coordinates, with $I = 1, \ldots, s-k$. If B is the constrained manifold, the solution of the non-holonomic system is the following section of $\mathcal{T}B$:

$$X = \mathbf{w}^I \mathcal{X}'_I + \dot{\mathbf{w}}^I \mathcal{V}'_I,$$

where $\{\mathcal{X}'_\alpha, \mathcal{V}'_\alpha\}$ is given by (10), that is,

$$X(a) = \left(a, \mathbf{w}^I f_I(q), \mathbf{w}^I X_I(a) + \dot{\mathbf{w}}^I \left.\frac{\partial}{\partial \mathbf{w}^I}\right|_a\right),$$

for all $a \in A_q$, where $X_I = \widetilde{\Psi}_{\beta I} \rho^i_\beta \frac{\partial}{\partial q^i}$. Therefore, the dynamics is given by the integral curves of the vector field on B,

$$\rho_{\mathcal{T}A}(X) = \mathbf{w}^I X_I + \dot{\mathbf{w}}^I \frac{\partial}{\partial \mathbf{w}^I}.$$

Example 6.2. Let $A = T\mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^2$ be a vector bundle, with local coordinates $(x, y, \dot{x}, \dot{y}, \omega_x, \omega_y, \omega_z)$ and $\{e_1 = (\partial_x, 0), e_2 = (\partial_y, 0), e_3 = (0, X_3), e_4 = (0, X_4), e_5 = (0, X_5)\}$ the associated local basis of sections of A. The vector bundle A can be endowed with a Lie algebroid structure $(\rho, [\cdot,\cdot]_A)$, locally given by

$$[e_3, e_4]_A = -e_5, \qquad [e_3, e_5]_A = e_4, \qquad [e_4, e_5]_A = -e_3,$$
$$\rho(e_1) = \frac{\partial}{\partial x}, \qquad \rho(e_2) = \frac{\partial}{\partial y}$$

and with the remaining structure functions being zero. Let us suppose that a non-holonomic system on the Lie algebroid A is characterized by the regular Lagrangian

$$\mathcal{L} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{k^2}{2}\left(\omega_x^2 + \omega_y^2 + \omega_z^2\right),$$

and the constraints given by

$$\phi_1 = \dot{x} - r\omega_y, \qquad \phi_2 = \dot{y} + r\omega_x,$$

where k, r are constants (see [7]). The solution of the system without constraints is a section of $\mathcal{T}A$ that is written in local coordinates as

$$X_{\mathcal{L}} = \dot{x}\mathcal{X}_1 + \dot{y}\mathcal{X}_2 + \omega_x\mathcal{X}_3 + \omega_y\mathcal{X}_4 + \omega_z\mathcal{X}_5,$$

where $\mathcal{X}_1(a) = \left(a, e_1(q), \left.\frac{\partial}{\partial x}\right|_a\right)$, $\mathcal{X}_2(a) = \left(a, e_2(q), \left.\frac{\partial}{\partial y}\right|_a\right)$ and $\mathcal{X}_i(a) = (a, e_i(q), 0)$, for all $a \in A_q$ and $i = 3, 4, 5$. Thus, Gibbs–Appell's function associated with $\mathcal{L}$ is given by

$$G_{\mathcal{L}} = \frac{1}{2}[(\dot{\mathbf{v}}^1)^2 + (\dot{\mathbf{v}}^2)^2] + \frac{k^2}{2}[(\dot{\mathbf{v}}^3)^2 + (\dot{\mathbf{v}}^4)^2 + (\dot{\mathbf{v}}^5)^2].$$

Let us consider the following coordinates:

$$\mathbf{w}^1 = \mathbf{v}^1 = \dot{x}, \qquad \mathbf{w}^2 = \mathbf{v}^2 = \dot{y}, \qquad \mathbf{w}^3 = \mathbf{v}^5 = \omega_z,$$
$$\mathbf{w}^4 = \phi_1 = \mathbf{v}^1 - r\mathbf{v}^4, \qquad \mathbf{w}^5 = \phi_2 = \mathbf{v}^2 + r\mathbf{v}^3.$$

Taking both the constraints and their time derivatives equal to zero, we obtain

$$G_{\mathcal{L}} = \frac{1}{2}[(\dot{\mathbf{w}}^1)^2 + (\dot{\mathbf{w}}^2)^2] + \frac{k^2}{2}\left[\left(\frac{-\dot{\mathbf{w}}^2}{r}\right)^2 + \left(\frac{\dot{\mathbf{w}}^1}{r}\right)^2 + (\dot{\mathbf{w}}^3)^2\right],$$

and solving the Gibbs–Appell equations, we arrive to

$$\dot{\mathbf{w}}^1 = \dot{\mathbf{w}}^2 = \dot{\mathbf{w}}^3 = 0,$$

i.e. the solution of this system with non-holonomic constraints is given by

$$X = \mathbf{w}^1\mathcal{X}'_1 + \mathbf{w}^2\mathcal{X}'_2 + \mathbf{w}^3\mathcal{X}'_3,$$

that is,

$$X(a) = \left(a, \dot{x}f_1(q) + \dot{y}f_2(q) + \omega_z f_3(q), \dot{x}\frac{\partial}{\partial x}\bigg|_a + \dot{y}\frac{\partial}{\partial y}\bigg|_a\right),$$

for all $a \in A_q$.

7. Conclusions

Throughout this paper we put in evidence the importance of the Lie algebroid formalism as a geometrical tool to deal with some problems in classical mechanics, mainly when we use quasi-coordinates instead of the usual coordinates. In fact, Lie algebroid theory provides a suitable geometric framework for dealing with quasi-coordinates. The meaning of Hamel symbols, for example, becomes clear from this new perspective. We believe that the geometrical description we have presented here, simplifies the computation of the equations and the solution of different problems using quasi-coordinates on a tangent bundle, particularly in the presence of non-holonomic constraints.

It has also been shown that the Lie algebroid formalism can be used to study systems with linear non-holonomic constraints as in the case of systems in classical mechanics on a tangent bundle. Again, the role of 'quasi-coordinates' is essential to solve these systems. The Gibbs–Appell generalized method is a useful tool to determine the equations of motion of a system with constraints. In most cases, this method is more efficient to solve systems with linear non-holonomic constraints than the Lagrange multipliers (generalized) method or even the computation of the generalized Euler–Lagrange equations, as was shown in several examples developed along the paper.

Acknowledgments

The authors acknowledge the referees for their comments. Financial support from CMUC/FCT, POCI/MAT/58452/2004, DGA-GRUPOS CONSOLIDADOS E24/1 and BFM-2003-02532 projects is acknowledged.

References

[1] Bates L and Śniatycki J 1993 Nonholonomic reduction *Rep. Math. Phys.* **32** 99–115

[2] Cariñena J F and Rañada M F 1993 Lagrangian systems with constraints: a geometric approach to the method of Lagrange multipliers *J. Phys. A: Math. Gen.* **26** 1335–51

[3] Cariñena J F, Nunes da Costa J M and Santos Patrícia 2005 Reduction of Lie algebroids structures *Int. J. Geom. Methods Mod. Phys.* **2** 965–91

[4] Cariñena J F, Nunes da Costa J M and Santos Patrícia 2006 Quasi-coordinates and Lagrangian mechanics on Lie algebroids *Proc. 14th Fall Workshop on Geometry and Physics (Bilbao 2005) (Publ. R. Soc. Mat. Esp.* vol 10*)* 249–54

[5] Cariñena J F, Nunes da Costa J M and Santos Patrícia 2007 Reduction of Lagrangian mechanics on Lie algebroids *J. Geom. Phys.* **57** 977–90
[6] Cortés J and Martínez E 2004 Mechanical control systems on Lie algebroids *IMA J. Math. Control Inf.* **21** 457–92
[7] Cortés J, de Léon M, Marrero J C and Martínez E 2005 Nonholonomic Lagrangian systems on Lie algebroids *Preprint* math-ph/0512003
[8] Crampin M 1981 On the differential geometry of the Euler–Lagrange equations, and the inverse problem of Lagrangian dynamics *J. Phys. A: Math. Gen.* **14** 2567–75
[9] Crampin M 1983 Tangent bundle geometry for Lagrangian dynamics *J. Phys. A: Math. Gen.* **16** 3755–72
[10] Crampin M and Pirani F A E 1986 *Applicable Differential Geometry* (Cambridge: Cambridge University Press)
[11] Desloge E A 1982 *Classical Mechanics* vol 2 (New York: Wiley)
[12] Gibbs J W 1879 On the fundamental formulae of dynamics *Am. J. Math.* **2** 49–64
[13] Higgins P J and Mackenzie K 1990 Algebraic constructions in the category of Lie algebroids *J. Algebra* **129** 194–230
[14] Koiller J 1991 Reduction of some classical non-holonomic systems with symmetry *Arch. Ration. Mech. Anal.* **118** 113–48
[15] Kotov A, Schaller P and Strobl T 2005 Dirac sigma models *Commun. Math. Phys.* **260** 455–80
[16] de León M, Marrero J C and Martínez E 2005 Lagrangian submanifolds and dynamics on Lie algebroids *J. Phys. A: Math. Gen.* **38** R241–308
[17] Lewis A D 1996 The geometry of the Gibbs–Appell equations and Gauss' principle of least constraint *Rep. Math. Phys.* **38** 11–28
[18] Marle C-M 1998 Various approaches to conservative and nonconservative noholonomics systems *Rep. Math. Phys.* **42** 211–29
[19] Martínez E 2001 Lagrangian mechanics on Lie algebroids *Acta Appl. Math.* **67** 295–320
[20] Mestdag T and Langerock B 2005 A Lie algebroid framework for non-holonomic systems *J. Phys. A: Math. Gen.* **38** 1097–111
[21] Mestdag T 2005 Lagrangian reduction by stages for non-holonomic systems in a Lie algebroid framework *J. Phys. A: Math. Gen.* **38** 10157–79
[22] Olshanetsky M A 2002 Lie algebroids as gauge symmetries in topological field theory *Multiple Facets of Quantization and Supersymmetry* ed M Olshanetsky and A Vainshtein (Singapore: World Scientific)
[23] Papastavridis J G 2002 *Analytical Mechanics* (Oxford: Oxford University Press)
[24] Pradines J 1967 Théorie de Lie pour les groupoïdes différentiables. Calcul différentiel dans la catégorie des groupoïdes infinitésimaux *C. R. Acad. Sci. Paris* A **264** 245–8
[25] Rosenberg R 1977 *Analytical Dynamics of Discrete Systems* (New York: Plenum)
[26] Saunders D J 1999 The geometry of non-holonomic Lagrangian systems *Satellite Conference of ICM in Berlin (Masaryk University in Brno)* 575–9
[27] Strobl T 2004 Gravity from Lie algebroid morphisms *Commun. Math. Phys.* **246** 475–502
[28] Weinstein A 1996 Lagrangian mechanics and groupoids *Fields Inst. Commun.* **7** 207–31
[29] Whittaker E T 1937 *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* (Cambridge: Cambridge University Press)